

THE DERIVED FUNCTORS OF UNRAMIFIED COHOMOLOGY

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ABSTRACT. We study the first “derived functors of unramified cohomology” in the sense of [14], applied to the sheaves \mathbb{G}_m and \mathcal{K}_2 . We find interesting connections with classical cycle-theoretic invariants of smooth projective varieties, involving notably a version of the Griffiths group, and the indecomposable $(2, 1)$ -cycles.

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INTRODUCTION

To a perfect field F , Voevodsky associates in [23] a *triangulated category of (bounded above) effective motivic complexes* $\mathbf{DM}_-^{\text{eff}}(F) = \mathbf{DM}_-^{\text{eff}}$. In [14], we rather worked with the unbounded version \mathbf{DM}^{eff} . We introduced a *triangulated category of birational motivic complexes*

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\mathbf{DM}° , and constructed a triple of adjoint functors

$$\mathbf{DM}^{\text{eff}} \begin{array}{c} \xrightarrow{R_{\text{nr}}} \\ \xleftarrow{i^\circ} \\ \xrightarrow{\nu_{\leq 0}} \end{array} \mathbf{DM}^\circ$$

with i° fully faithful. Via i° , the homotopy t -structure of \mathbf{DM}^{eff} induces a t -structure on \mathbf{DM}° (also called the homotopy t -structure), and the functors $\nu_{\leq 0}$, i° and R_{nr} are respectively right exact, exact and left exact.

The heart of \mathbf{DM}^{eff} is the abelian category \mathbf{HI} of *homotopy invariant Nisnevich sheaves with transfers* (see [23]). The heart of \mathbf{DM}° is the thick subcategory $\mathbf{HI}^\circ \subset \mathbf{HI}$ of *birational sheaves*: an object $\mathcal{F} \in \mathbf{HI}$ lies in \mathbf{HI}° if and only if it is locally constant for the Zariski topology.

In [14] we also started to study the right adjoint R_{nr} . Let $R_{\text{nr}}^0 = \mathcal{H}^0 \circ R_{\text{nr}} : \mathbf{HI} \rightarrow \mathbf{HI}^\circ$ be the induced functor. We proved that R_{nr}^0 is given by the formula $R_{\text{nr}}^0 \mathcal{F} = \mathcal{F}_{\text{nr}}$, where for a homotopy invariant sheaf $\mathcal{F} \in \mathbf{HI}$, \mathcal{F}_{nr} is defined by

$$(1) \quad \mathcal{F}_{\text{nr}}(X) = \text{Ker} \left(\mathcal{F}(K) \rightarrow \prod_v \mathcal{F}_{-1}(F(v)) \right).$$

Here X is a smooth connected F -variety, K is its function field, v runs through all divisorial discrete valuations on K trivial on F , with residue field $F(v)$, and \mathcal{F}_{-1} denotes the contraction of \mathcal{F} (see [22] or [17, Lect. 23]). Thus $R_{\text{nr}}^0 \mathcal{F}$ is the *unramified part* of \mathcal{F} .

Here is the example which connects the above to the classical situation of unramified cohomology. Let $i \geq 0$, $n \in \mathbf{Z}$ and let m be an integer invertible in F . Then the Nisnevich sheaf $\mathcal{F} = \mathcal{H}_{\text{ét}}^i(\mu_m^{\otimes n})$ associated to the presheaf

$$U \mapsto H_{\text{ét}}^i(U, \mu_m^{\otimes n})$$

defines an object of \mathbf{HI} , and $R_{\text{nr}}^0 \mathcal{F}$ is the usual unramified cohomology [6].

But the functor R_{nr} contains more information: for a general sheaf $\mathcal{F} \in \mathbf{HI}$, the birational sheaves

$$R_{\text{nr}}^q \mathcal{F} = \mathcal{H}^q(R_{\text{nr}} \mathcal{F})$$

need not be 0 for $q > 0$. Can we compute them, at least in some cases?

In this paper, we try our hand at the simplest examples: $\mathcal{F} = \mathbb{G}_m (= \mathcal{K}_1)$ and $\mathcal{F} = \mathcal{K}_2$. We cannot compute explicitly further than $q = 2$, except for varieties of dimension ≤ 2 ; but this already yields interesting connections with other birational invariants. For simplicity, *we restrict to the case where F is algebraically closed*; throughout this paper, the

cohomology we use is Nisnevich cohomology. The main results are the following:

Theorem 1. *Let X be a connected smooth projective F -variety. Then*

- (i) $R_{\text{nr}}^0 \mathbb{G}_m(X) = F^*$.
- (ii) $R_{\text{nr}}^1 \mathbb{G}_m(X) \xrightarrow{\sim} \text{Pic}^\tau(X)$.
- (iii) *There is a short exact sequence*

$$0 \rightarrow D^1(X) \rightarrow R_{\text{nr}}^2 \mathbb{G}_m(X) \rightarrow \text{Hom}(\text{Griff}_1(X), \mathbf{Z}) \rightarrow 0.$$
- (iv) *For $q \geq 3$, we have short exact sequences*

$$(0.1) \quad 0 \rightarrow \text{Ext}_{\mathbf{Z}}(\text{NS}_1(X, q-3), \mathbf{Z}) \rightarrow R_{\text{nr}}^q \mathbb{G}_m(X) \rightarrow \text{Hom}_{\mathbf{Z}}(\text{NS}_1(X, q-2), \mathbf{Z}) \rightarrow 0.$$

Here the notation is as follows: $\text{Pic}^\tau(X)$ is the group of cycle classes in $\text{Pic}(X) = CH^1(X)$ which are numerically equivalent to 0. We write $\text{Griff}_1(X) = \text{Ker} \left(A_1^{\text{alg}}(X) \rightarrow N_1(X) \right)$, where $A_1^{\text{alg}}(X)$ (resp. $N_1(X)$) denotes the group of 1-cycles on X modulo algebraic (resp. numerical) equivalence, and

$$D^1(X) = \text{Coker} \left(N^1(X) \rightarrow \text{Hom}(N_1(X), \mathbf{Z}) \right)$$

where $N^1(X) = \text{Pic}(X) / \text{Pic}^\tau(X)$ and the map is induced by the intersection pairing. Finally, the groups $\text{NS}_1(X, r)$ are those defined by Ayoub and Barbieri-Viale in [1, 3.25].

Note that $D^1(X)$ is a finite group since $N_1(X)$ is finitely generated.

After Colliot-Thélène complained that there was no unramified Brauer group in sight, we tried to invoke it by considering

$$\mathbb{G}_m^{\text{ét}} = R\alpha_* \alpha^* \mathbb{G}_m$$

where α is the projection of the étale site on smooth k -varieties onto the corresponding Nisnevich site. There is a natural map $\mathbb{G}_m \rightarrow \mathbb{G}_m^{\text{ét}}$, and

Theorem 2. *The map $R_{\text{nr}}^q \mathbb{G}_m \rightarrow R_{\text{nr}}^q \mathbb{G}_m^{\text{ét}}$ is an isomorphism for $q \leq 1$, and for $q = 2$ there is an exact sequence for any smooth projective X :*

$$0 \rightarrow R_{\text{nr}}^2 \mathbb{G}_m(X) \rightarrow R_{\text{nr}}^2 \mathbb{G}_m^{\text{ét}}(X) \rightarrow \text{Br}(X).$$

Considering now $\mathcal{F} = \mathcal{K}_2$:

Theorem 3. *We have an exact sequence*

$$\begin{aligned} 0 \rightarrow \text{Pic}^\tau(X) F^* &\rightarrow (R_{\text{nr}}^1 \mathcal{K}_2)(X) \rightarrow H_{\text{ind}}^1(X, \mathcal{K}_2) \\ &\xrightarrow{\bar{\delta}} \text{Hom}(\text{Griff}_1(X), F^*) \rightarrow (R_{\text{nr}}^2 \mathcal{K}_2)(X) \rightarrow CH^2(X) \end{aligned}$$

for any smooth projective variety X . Here

$$H_{\text{ind}}^1(X, \mathcal{K}_2) = \text{Coker}(\text{Pic}(X) \otimes F^* \rightarrow H^1(X, \mathcal{K}_2))$$

and

$$\text{Pic}^\tau(X)F^* = \text{Im}(\text{Pic}^\tau(X) \otimes F^* \rightarrow H^1(X, \mathcal{K}_2)).$$

The group $H^1(X, \mathcal{K}_2)$ appears in other guises, as the higher Chow group $CH^2(X, 1)$ or as the motivic cohomology group $H^3(X, \mathbf{Z}(2))$; its quotient $H_{\text{ind}}^1(X, \mathcal{K}_2)$ has been much studied and is known to be often nonzero. Note that, while it is not clear from the literature whether there exist smooth projective varieties X such that $\text{Hom}(\text{Griff}_1(X), \mathbf{Z}) \neq 0$, no such issue arises for $\text{Hom}(\text{Griff}_1(X), F^*)$ since F^* is divisible.

The following theorem was suggested by James Lewis. For a prime $l \neq \text{char } F$, write m_l for the exponent of the torsion subgroup of the l -adic cohomology group $H^3(X, \mathbf{Z}_l)$. Then $m_l = 1$ for almost all l : in characteristic 0 this follows from comparison with Betti cohomology, and in characteristic > 0 it is a famous theorem of Gabber [9]. Set $m = \text{lcm}(m_l)$: in characteristic 0, m is the exponent of $H_B^3(X, \mathbf{Z})_{\text{tors}}$, where H_B^* denotes Betti cohomology.

Theorem 4. *Assume that homological equivalence equals numerical equivalence on $CH_1(X) \otimes \mathbf{Q}$. Then, $m\bar{\delta} = 0$ in Theorem 3.*

0.1. **Remark.** 1) This hypothesis holds if $\text{char } F = 0$ by Lieberman [16, Cor. 1]. His argument shows that, in characteristic p , it holds for l -adic cohomology if and only if the Tate conjecture holds for divisors on X – more correctly for divisors on a model of X over a finitely generated field. In particular, it holds if X is an abelian variety; in this case, $m = 1$.

2) The prime to the characteristic part of the unramified Brauer group also appears in the exact sequence of Theorem 3 as a Tate twist of the torsion of $H_{\text{ind}}^1(X, \mathcal{K}_2)$ [13, Th. 1].

Theorem 5. *Suppose $\dim X \leq 2$ in Theorems 1 and 3. Then*

- (i) $R_{\text{nr}}^2 \mathbb{G}_m(X) \simeq D^1(X)$, $R_{\text{nr}}^3 \mathbb{G}_m(X) \simeq \text{Ext}_{\mathbf{Z}}(A_1^{\text{alg}}(X), \mathbf{Z}) \simeq \text{Hom}_{\mathbf{Z}}(\text{NS}(X)_{\text{tors}}, \mathbf{Q}/\mathbf{Z})$ and there exists an integer $m > 0$ such that $mR_{\text{nr}}^q \mathbb{G}_m(X) = 0$ for $q > 3$.
- (ii) *There exists an integer $m > 0$ such that $mR_{\text{nr}}^q \mathcal{K}_2(X) = 0$ for $q > 2$. Moreover, if $\dim X = 2$, the last map of Theorem 3 identifies $mR_{\text{nr}}^2(\mathcal{K}_2(X))$ with the Albanese kernel.*

Let $CH^2(X)_{\text{alg}}$ denote the subgroup of $CH^2(X)$ consisting of cycle classes algebraically equivalent to 0. Recall Murre's higher Abel-Jacobi map

$$AJ^3 : CH^2(X)_{\text{alg}} \rightarrow J^3(X)$$

where $J^3(X)$ is an algebraic intermediate Jacobian of X [18]. Theorem 5 (ii) suggests that in general, $\text{Im}(R_{\text{nr}}^2(\mathcal{K}_2)(X) \rightarrow CH^2(X))$ is contained in $\text{Ker } AJ^3$.

A key ingredient in the proofs of Theorems 1 and 3 is the work of Ayoub and Barbieri-Viale [1], which identifies the “maximal 0-dimensional quotient” of the Nisnevich sheaf (with transfers) associated to the presheaf $U \mapsto CH^n(X \times U)$ with the group $A_{\text{alg}}^n(X)$ of cycles modulo algebraic equivalence (see (5.3)).

The example $\mathcal{F} = \mathcal{H}_{\text{ét}}^i(\mu_m^{\otimes i})$ considered at the beginning relates to the sheaves studied in Theorems 1 and 3 through the Bloch-Kato conjecture: Kummer theory for \mathcal{K}_1 and the Merkurjev-Suslin theorem for \mathcal{K}_2 . Unfortunately, Theorem 1 barely suffices to compute $R_{\text{nr}}^q(\mathbb{G}_m/m)$ for $q \leq 1$ and we have not been able to deduce from Theorem 3 any meaningful information on $R_{\text{nr}}^*(\mathcal{K}_2/m)$. We give the result for $R_{\text{nr}}^1(\mathbb{G}_m/m)$ without proof; there is an exact sequence, where $\text{NS}(X)$ is the Néron-Severi group of X :

$$0 \rightarrow (\text{NS}(X)_{\text{tors}})/m \rightarrow R_{\text{nr}}^1(\mathbb{G}_m/m) \rightarrow {}_mD^1(X) \rightarrow 0$$

and encourage the reader to test his or her insight on this issue.

Let us end this introduction by a comment on the content of the statement “the assignment $X \mapsto \mathcal{F}(X)$ makes \mathcal{F} an object of \mathbf{HI}^o ”, which applies to the objects appearing in Theorems 1 and 3. It implies of course that $\mathcal{F}(X)$ is a (stable) birational invariant of smooth projective varieties, which was already known in most cases; but it also implies some non-trivial functoriality, due to the additional structure of presheaf with transfers on \mathcal{F} . For example, it yields a contravariant map $i^* : \mathcal{F}(X) \rightarrow \mathcal{F}(Y)$ for any closed immersion $i : Y \rightarrow X$. This does not seem easy to prove *a priori*, say for $\mathcal{F}(X) = D^1(X) = R_{\text{nr}}^2 \mathbb{G}_m(X)_{\text{tors}}$ in Theorem 1 (iii).

1. SOME RESULTS ON BIRATIONAL MOTIVES

We recall here some results from [14].

1.1. Lemma. *For any birational sheaf $\mathcal{F} \in \mathbf{HI}^o$ and any smooth variety X , $H^q(X, \mathcal{F}) = 0$ for $q > 0$.*

Proof. See [14, Prop. 1.3.3 b)]. □

For the next proposition, let us write

$$(1.1) \quad \nu^{\geq 1} M := \underline{\text{Hom}}(\mathbf{Z}(1), M)(1)$$

for $M \in \mathbf{DM}^{\text{eff}}$, where $\underline{\text{Hom}}$ is the internal Hom [23, Prop. 3.2.8].

1.2. Proposition. *For M as above, we have a functorial exact triangle*

$$\nu^{\geq 1} M \rightarrow M \rightarrow i^{\circ} \nu_{\leq 0} M \xrightarrow{+1}.$$

Moreover, $M \in \text{Im } i^{\circ}$ if and only if $\underline{\text{Hom}}(\mathbf{Z}(1), M) = 0$.

Proof. See [14, Prop. 3.6.2 and Lemma 3.5.4]. \square

1.3. Proposition. *For any $\mathcal{F} \in \mathbf{HI}$, the counit map*

$$i^{\circ} R_{\text{nr}}^0 \mathcal{F} \rightarrow \mathcal{F}$$

is a monomorphism.

Proof. See [14, Prop. 1.6.3]. \square

1.4. Proposition. *Let $\mathcal{F} \in \mathbf{HI}$. Then $\mathcal{F} \in \mathbf{HI}^0$ if and only if $\mathcal{F}_{-1} = 0$, where \mathcal{F}_{-1} is the contraction of \mathcal{F} ([22] or [17, Lect. 23]).*

Proof. This is [14, Prop. 1.5.2]. \square

1.5. Proposition. *Let $C \in \mathbf{DM}^{\text{eff}}$, and let $D = \underline{\text{Hom}}(\mathbf{Z}(1)[1], C)$. Then*

$$\mathcal{H}^i(D) = \mathcal{H}^i(C)_{-1}$$

for any $i \in \mathbf{Z}$.

Proof. This is [14, (4.1)]. \square

2. COMPUTATIONAL TOOLS

For $q \geq 0$, the R_{nr}^q 's define a cohomological δ -functor from \mathbf{HI} to \mathbf{HI}^0 . Since \mathbf{HI} is a Grothendieck category (it has a set of generators and exact filtering direct limits), it has enough injectives, so it makes sense to wonder if R_{nr}^q is the q -th derived functor of R_{nr}^0 . However, if $\mathcal{I} \in \mathbf{HI}$ is injective, while $R_{\text{nr}}^0 \mathcal{I}$ is clearly injective in \mathbf{HI}^0 , it is not clear whether $R_{\text{nr}}^q \mathcal{I} = 0$ for $q > 0$: the problem is similar to the one raised in [23, Rk. 1 after Prop. 3.1.8]. (In particular, the title of this paper should be taken with a pinch of salt.) Thus one cannot *a priori* compute the higher R_{nr}^q 's via injective resolutions; we give here another approach.

2.1. Lemma. *Let $\mathcal{F} \in \mathbf{HI}$, and let X be a smooth variety. Then the hypercohomology spectral sequence*

$$E_2^{p,q} = H^p(X, R_{\text{nr}}^q \mathcal{F}) \Rightarrow H^{p+q}(X, R_{\text{nr}} \mathcal{F})$$

degenerates, yielding isomorphisms

$$H^n(X, R_{\text{nr}} \mathcal{F}) \simeq H^n(X, R_{\text{nr}}^n \mathcal{F}).$$

Proof. Indeed, $E_2^{p,q} = 0$ for $p > 0$ by Lemma 1.1. \square

2.2. Proposition. *Let $C \in \mathbf{DM}^{\text{eff}}$, and let X be a smooth variety. Then we have a long exact sequence*

$$\begin{aligned} \cdots \rightarrow H^n(X, R_{\text{nr}}C) &\rightarrow H^n(X, C) \\ &\rightarrow \mathbf{DM}^{\text{eff}}(\nu^{\geq 1}M(X), C[n]) \rightarrow H^{n+1}(X, R_{\text{nr}}C) \rightarrow \cdots \end{aligned}$$

In particular, if $C = \mathcal{F}[0]$ for $\mathcal{F} \in \mathbf{HI}$, we get a long exact sequence

$$\begin{aligned} 0 \rightarrow R_{\text{nr}}^0\mathcal{F}(X) &\rightarrow \mathcal{F}(X) \rightarrow \mathbf{DM}^{\text{eff}}(\nu^{\geq 1}M(X), \mathcal{F}[0]) \rightarrow \cdots \\ &\rightarrow R_{\text{nr}}^n\mathcal{F}(X) \rightarrow H^n(X, \mathcal{F}) \rightarrow \mathbf{DM}^{\text{eff}}(\nu^{\geq 1}M(X), \mathcal{F}[n]) \rightarrow \cdots \end{aligned}$$

Proof. By iterated adjunction, we have

$$\begin{aligned} H^n(X, R_{\text{nr}}C) &\simeq \mathbf{DM}^{\text{eff}}(M(X), i^\circ R_{\text{nr}}C[n]) \\ &\simeq \mathbf{DM}^\circ(\nu_{\leq 0}M(X), R_{\text{nr}}C[n]) \simeq \mathbf{DM}^{\text{eff}}(i^\circ \nu_{\leq 0}M(X), C[n]). \end{aligned}$$

The first exact sequence then follows from Proposition 1.2. The second follows from the first, Lemma 2.1 and Proposition 1.3 b). \square

2.3. Proposition. *Let X be smooth and proper, and let $n \geq 0$. Then*

$$\underline{\text{Hom}}(\mathbf{Z}(n)[2n], M(X)) \in (\mathbf{DM}^{\text{eff}})^{\leq 0}.$$

Moreover,

$$\mathcal{H}^0(\underline{\text{Hom}}(\mathbf{Z}(n)[2n], M(X))) = \underline{CH}_n(X)$$

with

$$\underline{CH}_n(X)(U) = CH_n(X_{F(U)})$$

for any smooth connected variety U . Similarly, we have

$$\underline{\text{Hom}}(M(X), \mathbf{Z}(n)[2n]) \in (\mathbf{DM}^{\text{eff}})^{\leq 0}$$

and

$$\mathcal{H}^0(\underline{\text{Hom}}(M(X), \mathbf{Z}(n)[2n])) = \underline{CH}^n(X)$$

with

$$\underline{CH}^n(X)(U) = CH^n(X_{F(U)}).$$

Proof. The first statement is [11, Th. 2.2]. The second is proven similarly. \square

2.4. Lemma. *Let $\mathcal{F} \in \mathbf{HI}^\circ$. Then $R_{\text{nr}}^q i^\circ \mathcal{F} = 0$ for $q > 0$.*

Proof. This is obvious from the adjunction isomorphism $\mathcal{F}[0] \xrightarrow{\sim} R_{\text{nr}} i^\circ \mathcal{F}[0]$. \square

3. VARIETIES OF DIMENSION ≤ 2

As in [23, §3.4], let $d_{\leq 0} \mathbf{DM}^{\text{eff}}$ be the localising subcategory of \mathbf{DM}^{eff} generated by motives of varieties of dimension 0: since F is algebraically closed, this category is equivalent to the derived category $D(\mathbf{Ab})$ of abelian groups [23, Prop. 3.4.1]. In [1, Cor. 2.3.3], Ayoub and Barbieri-Viale show that the inclusion functor

$$j : d_{\leq 0} \mathbf{DM}^{\text{eff}} \hookrightarrow \mathbf{DM}^{\text{eff}}$$

has a left adjoint $L\pi_0$.

3.1. Lemma. *a) For any smooth connected variety X , the structural map $X \rightarrow \text{Spec } F$ induces an isomorphism $L\pi_0 M(X) \xrightarrow{\sim} L\pi_0 \mathbf{Z} = \mathbf{Z}$.*

b) We have $L\pi_0 \mathbb{G}_m = 0$.

c) If C is a smooth projective curve with Jacobian J (viewed as an object of \mathbf{HI}), then $L\pi_0 J = 0$.

d) If A is an abelian variety, viewed as an object of \mathbf{HI} (cf. [21, Lemma 3.2] or [2, Lemma 1.3.2]), then $L\pi_0 A$ is torsion. Moreover, $L_0\pi_0(A) := H_0(L\pi_0(A)) = 0$.

Proof. a) By adjunction and Yoneda's lemma, we have to show that for any object $C \in D(\mathbf{Ab})$, the map

$$H_{\text{Nis}}^*(F, C) \rightarrow H_{\text{Nis}}^*(X, C)$$

is an isomorphism. This is well-known: by a hypercohomology spectral sequence, reduce to C being a single abelian group; then C is flasque (see [19, Lemma 1.40]).

b) follows from a), applied to $X = \mathbf{P}^1$ (note that $M(\mathbf{P}^1) \simeq \mathbf{Z} \oplus \mathbb{G}_m[1]$).

c) Let $\tilde{M}(C)$ be the fibre of the map $M(C) \rightarrow \mathbf{Z}$. By a), $L\pi \tilde{M}(C) = 0$. By [23, Th. 3.4.2], we have an exact triangle

$$\mathbb{G}_m[1] \rightarrow \tilde{M}(C) \rightarrow J[0] \xrightarrow{+1}$$

so the claim follows from b).

d) As is well-known, there exists a curve C with Jacobian J and a morphism $J \rightarrow A$ which is split up to isogeny. The first claim then follows from c).

Let \mathbf{NST} be the category of Nisnevich sheaves with transfers [23]. To see that $L_0\pi_0(A) = 0$, it is equivalent by adjunction to see that $\text{Hom}_{\mathbf{NST}}(A, \mathcal{F}) = 0$ for any constant $\mathcal{F} \in \mathbf{NST}$. We may identify \mathcal{F} with its value on any connected $X \in \mathbf{Sm}$. Let $f : A \rightarrow \mathcal{F}$ be a morphism in \mathbf{NST} . Evaluating it on $1_A \in A(A)$, we get an element $f(1_A) \in \mathcal{F}(A) = \mathcal{F}$. If $X \in \mathbf{Sm}$ is connected and $a \in A(X) = \text{Hom}_F(X, A)$, then $f(a) = a^*f(1_A) = f(1_A)$. So f is constant, and

since it is additive it must send 0 to 0. This proves that $f = 0$, and thus $\mathrm{Hom}_{\mathbf{NST}}(A, \mathcal{F}) = 0$. \square

3.2. Proposition. *Let X/F be a smooth projective variety of dimension ≤ 2 . Then there exists an integer $m = m(X) > 0$ such that $m \mathrm{NS}_1(X, i) = 0$ for $i > 0$.*

Proof. Recall that $\mathrm{NS}_1(X, i) := H_i(L\pi_0 \underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)))$ [1, Def. 3.2.5]. For simplicity, write $C_X = \underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X))$. We go case by case, using Poincaré duality as in [11, Lemma B.1]:

If $\dim X = 0$, then $X = \mathrm{Spec}$ and hence $M(X) \simeq \mathbf{Z}$ is a birational motive; therefore $C_X = 0$ (Proposition 1.2) and $L\pi_0 C_X = 0$.

If $\dim X = 1$, then Poincaré duality produces an isomorphism

$$C_X \simeq \underline{\mathrm{Hom}}(M(X), \mathbf{Z}) \simeq \mathbf{Z}[0].$$

Hence $L\pi_0 C_X = \mathbf{Z}[0]$.

Now suppose that X is a smooth projective surface. By Poincaré duality, we get an isomorphism

$$C_X \simeq \underline{\mathrm{Hom}}(M(X), \mathbf{Z}(1)[2]).$$

By evaluating the latter complex against a varying smooth variety, one computes its homology sheaves as $\mathrm{Pic}_{X/F}$ and \mathbb{G}_m in degrees 0 and 1 respectively and zero elsewhere. Hence we have an exact triangle¹

$$\mathbb{G}_m[1] \rightarrow C_X \rightarrow \mathrm{Pic}_{X/F}[0] \xrightarrow{+1}.$$

We have $L\pi_0 \mathbb{G}_m[1] = 0$ by Lemma 3.1 b). On the other hand, the representability of $\mathrm{Pic}_{X/F}$ yields an exact sequence

$$0 \rightarrow \mathrm{Pic}_{X/F}^0 \rightarrow \mathrm{Pic}_{X/F} \rightarrow \mathrm{NS}_{X/F} \rightarrow 0$$

where $\mathrm{Pic}_{X/F}^0$ is the Picard variety of X and $\mathrm{NS}_{X/F}$ is the (constant) sheaf of connected components of the group scheme $\mathrm{Pic}_{X/F}$. Hence an exact triangle

$$L\pi_0 \mathrm{Pic}_{X/F}^0 \rightarrow L\pi_0 \mathrm{Pic}_{X/F} \rightarrow L\pi_0 \mathrm{NS}_{X/F} \xrightarrow{+1}.$$

where $L\pi_0 \mathrm{NS}_{X/F} = \mathrm{NS}(X)$. By Lemma 3.1 d), $L\pi_0 \mathrm{Pic}_{X/F}^0$ is torsion, which concludes the proof. (The vanishing of $L_0 \pi_0 \mathrm{Pic}_{X/F}^0$ gives back the isomorphism $L_0 \pi_0 C_X \xrightarrow{\sim} \mathrm{NS}(X)$ of [1], see (5.3) below.) \square

¹It is split by the choice of a rational point of X , but this is useless for the proof.

4. BIRATIONAL MOTIVES AND INDECOMPOSABLE $(2, 1)$ -CYCLES

In this section, we only assume F perfect; we give proofs of two results promised in [14, Rks 3.6.4 and 3.4.2]. These results are not used in the rest of the paper.

For the first one, let X be a smooth projective variety, and let $M = \underline{\mathrm{Hom}}(M(X), \mathbf{Z}(2)[4])$. Note that $M \simeq M(X)$ if $\dim X = 2$ by Poincaré duality (cf. proof of Proposition 3.2). The functor $\nu_{\leq 0}$ is right t -exact as the left adjoint of the t -exact functor i^o [14, Th. 3.4.1], so $\nu_{\leq 0}M \in (\mathbf{DM}^o)^{\leq 0}$ since $M \in (\mathbf{DM}^{\mathrm{eff}})^{\leq 0}$ by Proposition 2.3. We want to compute the last two non-zero cohomology sheaves of $\nu_{\leq 0}M$. Here is the result:

4.1. Theorem. *With the above notation, we have*

$$\mathcal{H}^i(\nu_{\leq 0}M) = \begin{cases} \underline{CH}^2(X) & \text{for } i = 0 \\ \underline{H}_{\mathrm{ind}}^1(X, \mathcal{K}_2) & \text{for } i = -1 \end{cases}$$

where the sections of $\underline{H}_{\mathrm{ind}}^1(X, \mathcal{K}_2)$ over a smooth connected F -variety U with function field K are given by the formula

$$\underline{H}_{\mathrm{ind}}^1(X, \mathcal{K}_2)(U) = \mathrm{Coker} \left(\bigoplus_{[L:K] < \infty} \mathrm{Pic}(X_L) \otimes L^* \rightarrow H^1(X_K, \mathcal{K}_2) \right)$$

in which the map is given by products and transfers.

Proof. We use the exact triangle of Proposition 1.2. From the cancellation theorem ([24], [11, Prop. A.1]), we get an isomorphism

$$\nu^{\geq 1}M \simeq \underline{\mathrm{Hom}}(M(X), \mathbf{Z}(1)[4])(1) \simeq C_X \otimes \mathbb{G}_m[1]$$

where $C_X = \underline{\mathrm{Hom}}(M(X), \mathbf{Z}(1)[2])$.

By Proposition 2.3, $C_X \in (\mathbf{DM}^{\mathrm{eff}})^{\leq 0}$. On the other hand, \otimes is right t -exact because it is induced by a right t -exact \otimes -functor on $D(\mathbf{NST})$ via the right t -exact functor $LC : D(\mathbf{NST}) \rightarrow \mathbf{DM}^{\mathrm{eff}}$. Hence $\nu^{\geq 1}M \in (\mathbf{DM}^{\mathrm{eff}})^{\leq -1}$.

Using Proposition 2.3 again, this shows the assertion in the case $i = 0$ (compare [11, Th. 2.2 and its proof]). For the case $i = -1$, the long exact sequence of cohomology sheaves yields an exact sequence:

$$\cdots \rightarrow \mathcal{H}^0(C_X \otimes \mathbb{G}_m) \rightarrow \mathcal{H}^{-1}(M) \rightarrow \mathcal{H}^{-1}(i^o \nu_{\leq 0}M) \rightarrow 0.$$

Let $\mathcal{F} = \mathcal{H}^0(C_X) = \underline{CH}^1(X)$; then $\mathcal{H}^0(C_X \otimes \mathbb{G}_m) = \mathcal{F} \otimes_{\mathbf{HI}} \mathbb{G}_m$ by right t -exactness of \otimes ; here $\otimes_{\mathbf{HI}}$ is the tensor structure induced by \otimes

on **HI**. For any function field K/F , the map induced by transfers

$$\bigoplus_{[L:K] < \infty} \mathcal{F}(L) \otimes \mathbb{G}_m(L) \rightarrow (\mathcal{F} \otimes_{\mathbf{HI}} \mathbb{G}_m)(K)$$

is surjective [15, 2.14], which concludes the proof. \square

The second result which was promised in [14, Rk. 4.3.2] is:

4.2. Proposition. *Let E be an elliptic curve over F . Then the sheaf*

$$\mathrm{Tor}_1^{\mathbf{DM}}(E, E) := \mathcal{H}^{-1}(E[0] \otimes E[0])$$

*is not birational. Here E is viewed as an object of **HI** [2, Lemma 1.4.4].*

(This contrasts with the fact that the tensor product of two birational sheaves is birational, [14, Th. 4.3.1].)

Proof. Up to extending scalars, we may and do assume that $\mathrm{End}(E) = \mathrm{End}(E_{\bar{F}})$. Consider the surface $X = E \times E$. The choice of the rational point $0 \in E$ yields a Chow-Künneth decomposition of the Chow motive of E , hence by [23, Prop. 2.1.4] an isomorphism

$$M(E) \simeq \mathbf{Z}[0] \oplus E[0] \oplus \mathbf{Z}(1)[2]$$

(compare also [23, Th. 3.4.2]). Therefore

$$M(X) \simeq \mathbf{Z}[0] \oplus 2E[0] \oplus 2\mathbf{Z}(1)[2] \oplus E[0] \otimes E[0] \oplus 2E(1)[2] \oplus \mathbf{Z}(2)[4].$$

This allows us to compute $\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], E[0] \otimes E[0])$ as a direct summand of $\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)) = C_X$. First we have

$$\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], \mathbf{Z}[0]) = \underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], E[0]) = 0.$$

The first vanishing is [11, Lemma A.2], while the second one follows from the Poincaré duality isomorphism $\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(E)) \simeq \underline{\mathrm{Hom}}(M(E), \mathbf{Z}) = \mathbf{Z}$ [12, Lemma 2.1 a)]. Hence, using the cancellation theorem:

$$C_X \simeq 2\mathbf{Z}[0] \oplus \underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], E[0] \otimes E[0]) \oplus 2E[0] \oplus \mathbf{Z}(1)[2]$$

and

$$\mathrm{Pic}_{X/F} = \mathcal{H}^0(C_X) \simeq 2\mathbf{Z} \oplus \mathcal{H}^0(\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], E[0] \otimes E[0])) \oplus 2E.$$

On the other hand, using Weil's formula for the Picard group of a product, we have a canonical decomposition

$$\mathrm{Pic}_{E \times E/F} \simeq \mathrm{Pic}_{E \times E/F}^0 \oplus \mathrm{NS}(E) \oplus \mathrm{NS}(E) \oplus \mathrm{Hom}(E, E) = 2E \oplus 2\mathbf{Z} \oplus \mathrm{End}(E).$$

One checks that the idempotents involved in the two decompositions of $\mathrm{Pic}_{X/F}$ match to yield an isomorphism

$$\mathcal{H}^0(\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], E[0] \otimes E[0])) \simeq \mathrm{End}(E)$$

where $\text{End}(E)$ is viewed as a constant sheaf. By the t -exactness of Voevodsky's contraction functor $(-)_-1 = \underline{\text{Hom}}(\mathbb{G}_m, -)$ [14, Prop. 4.1.1], this yields an isomorphism $\text{End}(E) \xrightarrow{\sim} \text{Tor}_1^{\mathbf{DM}}(E, E)_{-1}$, which proves that $\text{Tor}_1^{\mathbf{DM}}(E, E)$ is not birational (see Proposition 1.4). \square

5. THE CASE OF \mathbb{G}_m : PROOF OF THEOREMS 1, 2 AND 5 (i)

5.A. Proof of Theorem 1. We apply Proposition 2.2 to $\mathcal{F} = \mathbb{G}_m$. The Nisnevich cohomology of \mathbb{G}_m is well-known: we have

$$H^n(X, \mathbb{G}_m) = \begin{cases} F^* & \text{if } n = 0 \\ \text{Pic}(X) & \text{if } n = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Noting that $\mathbb{G}_m[0] = \mathbf{Z}(1)[1]$ in \mathbf{DM}^{eff} , we get

$$\begin{aligned} \mathbf{DM}^{\text{eff}}(\nu^{\geq 1} M(X), \mathbb{G}_m[n]) &= \\ \mathbf{DM}^{\text{eff}}(\underline{\text{Hom}}(\mathbf{Z}(1), M(X))(1), \mathbf{Z}(1)[n+1]) &= \\ \mathbf{DM}^{\text{eff}}(\underline{\text{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbf{Z}[n-1]) \end{aligned}$$

by using the cancellation theorem. Thus

$$\begin{aligned} (5.1) \quad \mathbf{DM}^{\text{eff}}(\underline{\text{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbf{Z}[n-1]) \\ \simeq d_{\leq 0} \mathbf{DM}^{\text{eff}}(L\pi_0 \underline{\text{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbf{Z}[n-1]) \\ = D(\mathbf{Ab})(L\pi_0 \underline{\text{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbf{Z}[n-1]) =: F_n(X). \end{aligned}$$

The homology group $H_s(L\pi_0 \underline{\text{Hom}}(\mathbf{Z}(1)[2], M(X)))$ is denoted by $\text{NS}_1(X, s)$ in [1, 3.25]. The universal coefficient theorem then gives an exact sequence

$$\begin{aligned} (5.2) \quad 0 \rightarrow \text{Ext}_{\mathbf{Ab}}(\text{NS}_1(X, n-2), \mathbf{Z}) \rightarrow F_n(X) \\ \rightarrow \mathbf{Ab}(\text{NS}_1(X, n-1), \mathbf{Z}) \rightarrow 0. \end{aligned}$$

By Proposition 2.3, $\underline{\text{Hom}}(\mathbf{Z}(1)[2], M(X)) \in (\mathbf{DM}^{\text{eff}})^{\leq 0}$. Since the inclusion functor j is t -exact, $L\pi_0$ is right t -exact by a general result on triangulated categories [3, Prop. 1.3.17], hence $\text{NS}_1(X, n) = 0$ for $n < 0$. For $n = 0$, Ayoub and Barbieri-Viale find

$$(5.3) \quad \text{NS}_1(X, 0) = A_1^{\text{alg}}(X)$$

in [1, Th. 3.1.4]².

²The hypothesis F algebraically closed is sufficient for their proof.

Gathering all this, we get (i) (which also follows from (1)), an exact sequence

$$(5.4) \quad 0 \rightarrow R_{\text{nr}}^1 \mathbb{G}_m(X) \rightarrow \text{Pic}(X) \xrightarrow{\delta} \text{Hom}(A_1^{\text{alg}}(X), \mathbf{Z}) \rightarrow R_{\text{nr}}^2 \mathbb{G}_m(X) \rightarrow 0$$

and isomorphisms

$$(5.5) \quad F_n(X) \xrightarrow{\sim} R_{\text{nr}}^{n+1} \mathbb{G}_m(X)$$

for $n \geq 2$, which yield (iv) thanks to (5.4).

In Lemma 5.1 below, we shall show that δ is induced by the intersection pairing. Granting this for the moment, (ii) is immediate and we get a cross of exact sequences

$$\begin{array}{ccccccc} & & & 0 & & & \\ & & & \downarrow & & & \\ & & & \text{Hom}(N_1(X), \mathbf{Z}) & & & \\ & \nearrow & & \downarrow & & & \\ 0 & \longrightarrow & N^1(X) & \longrightarrow & \text{Hom}(A_1^{\text{alg}}(X), \mathbf{Z}) & \longrightarrow & R_{\text{nr}}^2 \mathbb{G}_m(X) \longrightarrow 0 \\ & & & & \downarrow & & \\ & & & & \text{Hom}(\text{Griff}_1(X), \mathbf{Z}) & & \\ & & & & \downarrow & & \\ & & & & 0 & & \end{array}$$

in which the triangle commutes, and where we used that $N_1(X)$ is a free finitely generated abelian group. The exact sequence of (iii) then follows from a diagram chase.

5.1. Lemma. *The map δ of (5.4) is induced by the intersection pairing.*

Proof. This map comes from the composition

$$\begin{aligned} (5.6) \quad & \mathbf{DM}^{\text{eff}}(M(X), \mathbf{Z}(1)[2]) \\ & \rightarrow \mathbf{DM}^{\text{eff}}(\underline{\text{Hom}}(\mathbf{Z}(1)[2], M(X))(1)[2], \mathbf{Z}(1)[2]) \\ & = \mathbf{DM}^{\text{eff}}(\underline{\text{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbf{Z}) \\ & \rightarrow \text{Hom}_{\mathbf{Z}}(\text{Hom}(\mathbf{Z}(1)[2], M(X)), \mathbf{Z}) \end{aligned}$$

in which the first map is induced by the canonical morphism $\nu^{\geq 1} M(X) \rightarrow M(X)$, the equality follows from the cancellation theorem [24] and the third is by taking global sections at $\text{Spec } k$.

Consider the natural pairing

$$\begin{aligned} \underline{\mathrm{Hom}}(M(X), \mathbf{Z}(1)[2]) \otimes \underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)) \\ \rightarrow \underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], \mathbf{Z}(1)[2]) = \mathbf{Z}[0]. \end{aligned}$$

By Proposition 2.3, this pairing factors through a pairing

$$\underline{CH}^1(X)[0] \otimes \underline{CH}_1(X)[0] \rightarrow \mathbf{Z}[0].$$

Taking global sections, we clearly get the intersection pairing.

From the above, we get a commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{Hom}}(M(X), \mathbf{Z}(1)[2]) & \longrightarrow & \underline{\mathrm{Hom}}(\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbf{Z}[0]) \\ \downarrow & & \uparrow \\ \underline{CH}^1(X)[0] & \longrightarrow & \underline{\mathrm{Hom}}(\underline{CH}_1(X)[0], \mathbf{Z}[0]). \end{array}$$

Applying the functor $\mathbf{DM}^{\mathrm{eff}}(\mathbf{Z}, -)$ to this diagram, we get a commutative diagram of abelian groups

$$\begin{array}{ccc} \mathbf{DM}^{\mathrm{eff}}(M(X), \mathbf{Z}(1)[2]) & \xrightarrow{a} & \mathbf{DM}^{\mathrm{eff}}(\underline{\mathrm{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbf{Z}[0]) \\ \downarrow & & \uparrow b \\ CH^1(X) & \longrightarrow & \mathbf{DM}^{\mathrm{eff}}(\underline{CH}_1(X)[0], \mathbf{Z}[0]). \end{array}$$

In this diagram, one checks easily that a corresponds to (5.6) via the cancellation theorem. On the other hand, b is an isomorphism. Now the evaluation map at $\mathrm{Spec} F$, $\mathcal{F} \mapsto \mathcal{F}(F)$, yields a commutative triangle

$$\begin{array}{ccc} CH^1(X) & \longrightarrow & \mathbf{DM}^{\mathrm{eff}}(\underline{CH}_1(X)[0], \mathbf{Z}[0]) \\ & \searrow \cap & \downarrow ev_F \\ & & \mathrm{Hom}(CH_1(X), \mathbf{Z}). \end{array}$$

where \cap is the intersection pairing (see above). But we saw that $\mathbf{DM}^{\mathrm{eff}}(\underline{CH}_1(X)[0], \mathbf{Z}[0]) \simeq \mathrm{Hom}(A_1^{\mathrm{alg}}(X), \mathbf{Z})$; via this isomorphism, ev_F is induced by the surjection $CH^1(X) \twoheadrightarrow A_{\mathrm{alg}}^1(X)$, hence is injective. This concludes the proof. \square

5.B. Proof of Theorem 2. We use the following lemma:

5.2. Lemma. *In $\mathbf{DM}^{\mathrm{eff}}$, the map $\mathbb{G}_m \rightarrow \mathbb{G}_m^{\mathrm{\acute{e}t}}$ is an isomorphism on H^0 ; moreover, $R^1\alpha_*\alpha^*\mathbb{G}_m = 0$ and $R^2\alpha_*\alpha^*\mathbb{G}_m$ is the Nisnevich sheaf Br associated to the presheaf $U \mapsto \mathrm{Br}(U)$ ³. Here, $\alpha : \mathbf{Sm}_{\mathrm{\acute{e}t}} \rightarrow \mathbf{Sm}_{\mathrm{Nis}}$ is the change of topology morphism.*

³This presheaf is in fact already a Nisnevich sheaf.

Proof. The first statement is obvious, the second one follows from the local vanishing of Pic and the third one is tautological. \square

To compute $R_{\text{nr}}\mathbb{G}_m^{\text{ét}}$, we may use the “hypercohomology” spectral sequence

$$E_2^{p,q} = R_{\text{nr}}^p R^q \alpha_* \alpha^* \mathbb{G}_m \Rightarrow R_{\text{nr}}^{p+q} \mathbb{G}_m^{\text{ét}}.$$

From Lemma 5.2, we find an isomorphism

$$R_{\text{nr}}^1 \mathbb{G}_m \xrightarrow{\sim} R_{\text{nr}}^1 \mathbb{G}_m^{\text{ét}}$$

and a five term exact sequence

$$0 \rightarrow R_{\text{nr}}^2 \mathbb{G}_m \rightarrow R_{\text{nr}}^2 \mathbb{G}_m^{\text{ét}} \rightarrow R_{\text{nr}}^0 \text{Br} \rightarrow R_{\text{nr}}^3 \mathbb{G}_m \rightarrow R_{\text{nr}}^3 \mathbb{G}_m^{\text{ét}}$$

which yields (a more precise form of) Theorem 2 in view of the obvious isomorphism $R_{\text{nr}}^0 \text{Br} = \text{Br}_{\text{nr}}$, where Br_{nr} is the unramified Brauer group.

5.C. Proof of Theorem 5 (i). Since $\dim X \leq 2$, $\text{Griff}_1(X)$ is torsion hence $\text{Hom}(\text{Griff}_1(X), \mathbf{Z}) = 0$, which gives the first statement. Then, Theorem 1 (iv) and Proposition 3.2 yield isomorphisms

$$\text{Ext}_{\mathbf{Z}}(\text{NS}_1(X, q-3), \mathbf{Z}) \xrightarrow{\sim} R_{\text{nr}}^q \mathbb{G}_m(X), \quad q \geq 3.$$

For $q > 3$, the left hand group is killed by the integer m of Proposition 3.2. Suppose $q = 3$; then $\text{NS}_1(X, q-3) = A_1^{\text{alg}}(X)$, which proves Theorem 5 (i) except for the isomorphism involving $\text{NS}(X)_{\text{tors}}$. For this we distinguish 3 cases:

- (1) If $\dim X = 0$, $A_1^{\text{alg}}(X) = \text{NS}(X) = 0$ and the statement is true.
- (2) If $\dim X = 1$, $A_1^{\text{alg}}(X) \simeq \mathbf{Z} \simeq \text{NS}(X)$ and the statement is still true.
- (3) If $\dim X = 2$, $A_1^{\text{alg}}(X) = \text{NS}(X)$. But for any finitely generated abelian group A , there is a string of canonical isomorphisms

$$\text{Ext}_{\mathbf{Z}}(A, \mathbf{Z}) \xrightarrow{\sim} \text{Ext}_{\mathbf{Z}}(A_{\text{tors}}, \mathbf{Z}) \xleftarrow{\sim} \text{Hom}_{\mathbf{Z}}(A_{\text{tors}}, \mathbf{Q}/\mathbf{Z}).$$

This concludes the proof.

6. THE CASE OF \mathcal{K}_2 : PROOF OF THEOREMS 3 AND 5 (ii)

6.1. Lemma. *a) The natural map*

$$(6.1) \quad \mathbf{Z}(2)[2] \rightarrow \mathcal{K}_2[0]$$

induces an isomorphism

$$\text{cone}(i^{\circ} R_{\text{nr}} \mathbf{Z}(2)[2] \rightarrow \mathbf{Z}(2)[2]) \xrightarrow{\sim} \text{cone}(i^{\circ} R_{\text{nr}} \mathcal{K}_2[0] \rightarrow \mathcal{K}_2[0]).$$

b) The map (6.1) induces an isomorphism

$$\mathbf{DM}^{\text{eff}}(\nu^{\geq 1} C, \mathbf{Z}(2)[2]) \xrightarrow{\sim} \mathbf{DM}^{\text{eff}}(\nu^{\geq 1} C, \mathcal{K}_2[0])$$

for any $C \in \mathbf{DM}^{\text{eff}}$. (See (1.1) for the definition of $\nu^{\geq 1}C$.)

Proof. By the cancellation theorem, we have

$$\underline{\text{Hom}}(\mathbf{Z}(1)[1], \mathbf{Z}(2)[2]) \simeq \mathbf{Z}(1)[1] \simeq \mathbb{G}_m[0]$$

in \mathbf{DM}^{eff} .

Let $\mathcal{H}^i(C)$ denote the i -th cohomology sheaf of an object $C \in \mathbf{DM}^{\text{eff}}$. By Proposition 1.5, the i -th cohomology sheaf of the left hand side is $\mathcal{H}^i(\mathbf{Z}(2)[2])_{-1}$. Thus the latter sheaf is 0 for $i \neq 0$. By Proposition 1.4, $\mathcal{H}^i(\mathbf{Z}(2)[2]) \in \mathbf{HI}^o$ for $i \neq 0$, hence $\tau_{<0}(\mathbf{Z}(2)[2]) \in \mathbf{DM}^o$. By adjunction, we deduce

$$\text{cone}(i^o R_{\text{nr}} \tau_{<0}(\mathbf{Z}(2)[2]) \rightarrow \tau_{<0}(\mathbf{Z}(2)[2])) = 0$$

which in turn implies a).

To pass from a) to b), use the fact that, for $C, D \in \mathbf{DM}^{\text{eff}}$, adjunction transforms the exact sequence

$$\mathbf{DM}^{\text{eff}}(i^o \nu_{\leq 0} C, D) \rightarrow \mathbf{DM}^{\text{eff}}(C, D) \rightarrow \mathbf{DM}^{\text{eff}}(\nu^{\geq 1} C, D)$$

into the exact sequence

$$\begin{aligned} \mathbf{DM}^{\text{eff}}(C, i^o R_{\text{nr}} D) &\rightarrow \mathbf{DM}^{\text{eff}}(C, D) \\ &\rightarrow \mathbf{DM}^{\text{eff}}(C, \text{cone}(i^o R_{\text{nr}} D \rightarrow D)). \end{aligned}$$

□

Proof of Theorem 3. Applying the exact sequence of Proposition 2.2 to $C = \mathcal{K}_2[0]$ and using Lemma 6.1 b), we get a long exact sequence

$$\begin{aligned} \cdots &\rightarrow H^n(X, R_{\text{nr}} \mathcal{K}_2) \rightarrow H^n(X, \mathcal{K}_2) \\ &\rightarrow \mathbf{DM}^{\text{eff}}(\nu^{\geq 1} M(X), \mathbf{Z}(2)[n+2]) \rightarrow H^{n+1}(X, R_{\text{nr}} \mathcal{K}_2) \rightarrow \cdots \end{aligned}$$

Using the cancellation theorem, we get an isomorphism

$$\mathbf{DM}^{\text{eff}}(\nu^{\geq 1} M(X), \mathbf{Z}(2)[n+2]) \simeq \mathbf{DM}^{\text{eff}}(\underline{\text{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbf{Z}(1)[n]).$$

Since $\mathbf{Z}(1)[n] = \mathbb{G}_m[n-1]$, using Lemma 2.1 we get an exact sequence

$$\begin{aligned} (6.2) \quad 0 &\rightarrow (R_{\text{nr}}^1 \mathcal{K}_2)(X) \rightarrow H^1(X, \mathcal{K}_2) \\ &\xrightarrow{\delta} \mathbf{DM}^{\text{eff}}(\underline{\text{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbb{G}_m[0]) \rightarrow (R_{\text{nr}}^2 \mathcal{K}_2)(X) \rightarrow CH^2(X) \end{aligned}$$

where we also used that $H^2(X, \mathcal{K}_2) \simeq CH^2(X)$.

The group $\mathbf{DM}^{\text{eff}}(\underline{\mathbf{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbb{G}_m[0])$ may be computed as follows:

$$\begin{aligned}
 (6.3) \quad \mathbf{DM}^{\text{eff}}(\underline{\mathbf{Hom}}(\mathbf{Z}(1)[2], M(X)), \mathbb{G}_m[0]) \\
 &\stackrel{1}{\simeq} \mathbf{HI}(\mathcal{H}_0(\underline{\mathbf{Hom}}(\mathbf{Z}(1)[2], M(X))), \mathbb{G}_m) \\
 &\stackrel{2}{\simeq} \mathbf{HI}(\underline{CH}_1(X), \mathbb{G}_m) \stackrel{3}{\simeq} \mathbf{HI}^0(\underline{CH}_1(X), R_{\text{nr}}^0 \mathbb{G}_m) \\
 &\stackrel{4}{\simeq} \mathbf{HI}(\underline{CH}_1(X), jF^*) \stackrel{5}{\simeq} \mathbf{Ab}(L_0 \pi_0 \underline{CH}_1(X), F^*) \\
 &\stackrel{6}{\simeq} \mathbf{Ab}(A_1^{\text{alg}}(X), F^*).
 \end{aligned}$$

Here, isomorphism 1 follows from the fact that $\underline{\mathbf{Hom}}(\mathbf{Z}(1)[2], M(X)) \in (\mathbf{DM}^{\text{eff}})^{\leq 0}$ (Proposition 2.3), 2 comes from the computation of \mathcal{H}_0 (ibid.), 3 follows from adjunction, knowing that $\underline{CH}_1(X)$ is a birational sheaf (ibid.), 4 follows from Theorem 1 (i), 5 comes from adjunction and 6 follows from (5.3).

Thus the homomorphism δ corresponds to a pairing

$$H^1(X, \mathcal{K}_2) \times A_1^{\text{alg}}(X) \rightarrow F^*.$$

Let $d = \dim X$. An argument analogous to that in the proof of Lemma 5.1 shows that this pairing comes from the “intersection” pairing

$$\begin{aligned}
 (6.4) \quad H^3(X, \mathbf{Z}(2)) \times H^{2d-2}(X, \mathbf{Z}(d-1)) &\xrightarrow{\cap} H^{2d+1}(X, \mathbf{Z}(d+1)) \\
 &\xrightarrow{\pi_*} H^1(F, \mathbf{Z}(1)) = F^*
 \end{aligned}$$

where the last map is induced by the “Gysin” morphism ${}^t\pi : \mathbf{Z}(d)[2d] \rightarrow M(X)$. Here we used the isomorphisms

$$H^1(X, \mathcal{K}_2) \simeq H^3(X, \mathbf{Z}(2)), \quad CH_1(X) \simeq H^{2d-2}(X, \mathbf{Z}(d-1)).$$

In particular, (6.4) factors through algebraic equivalence. This was proved by Coombes [8, Cor. 2.14] in the special case of a surface; we shall give a different proof below, which avoids the use of (5.3).

Consider the map

$$c : CH^1(X) \otimes F^* = H^1(X, \mathcal{K}_1) \otimes H^0(X, \mathcal{K}_1) \rightarrow H^1(X, \mathcal{K}_2).$$

By functoriality, we have a commutative diagram of pairings

$$\begin{array}{ccc}
 CH^1(X) \otimes F^* \times A_1^{\text{alg}}(X) & \longrightarrow & F^* \\
 c \times 1 \downarrow & & \parallel \\
 H^1(X, \mathcal{K}_2) \times A_1^{\text{alg}}(X) & \longrightarrow & F^*
 \end{array}$$

where the top pairing is the intersection pairing $CH^1(X) \times A_1^{\text{alg}}(X) \rightarrow \mathbf{Z}$, tensored with F^* . Since the latter is 0 when restricted to $\text{Griff}_1(X)$, we get an induced pairing

$$H_{\text{ind}}^1(X, \mathcal{K}_2) \times \text{Griff}_1(X) \rightarrow F^*$$

yielding a commutative diagram

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 0 \rightarrow \text{Pic}^\tau(X) \otimes F^* & \longrightarrow & \text{Pic}(X) \otimes F^* & \xrightarrow{\alpha} & \text{Hom}(A_1^{\text{num}}(X), F^*) & & \\
 & & \downarrow & & \downarrow & & \\
 0 \rightarrow (R_{\text{nr}}^1 \mathcal{K}_2)(X) & \longrightarrow & H^1(X, \mathcal{K}_2) & \xrightarrow{\delta} & \text{Hom}(A_1^{\text{alg}}(X), F^*) & & \\
 & & \downarrow & & \downarrow & & \\
 & & H_{\text{ind}}^1(X, \mathcal{K}_2) & \xrightarrow{\bar{\delta}} & \text{Hom}(\text{Griff}_1(X), F^*) & & \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

In this diagram, all rows and columns are complexes. The middle row and the two columns are exact; moreover, α is surjective as one sees by tensoring with F^* the exact sequence

$$0 \rightarrow \text{Pic}^\tau(X) \rightarrow \text{Pic}(X) \rightarrow \text{Hom}(A_1^{\text{num}}(X), \mathbf{Z}) \rightarrow D^1(X) \rightarrow 0.$$

Then a diagram chase yields an exact sequence

$$\text{Pic}^\tau(X) \otimes F^* \rightarrow (R_{\text{nr}}^1 \mathcal{K}_2)(X) \rightarrow H_{\text{ind}}^1(X, \mathcal{K}_2) \xrightarrow{\bar{\delta}} \text{Hom}(\text{Griff}_1(X), F^*)$$

and the surjectivity of α implies that the map $\text{Hom}(A_1^{\text{alg}}(X), F^*) \rightarrow (R_{\text{nr}}^2 \mathcal{K}_2)(X)$ given by (6.2) and (6.3) factors through $\text{Hom}(\text{Griff}_1(X), F^*)$. This concludes the proof. \square

Direct proof that (6.4) factors through algebraic equivalence. Consider classes $\alpha \in H^3(X, \mathbf{Z}(2))$ and $\beta \in CH^{d-1}(X)$: assuming that β is algebraically equivalent to 0, we must prove that $\pi_*(\alpha \cdot \beta) = 0$, where π is the projection $X \rightarrow \text{Spec } F$.

By hypothesis, there exists a smooth projective curve C , two points $c_0, c_1 \in C$ and a cycle class $\gamma \in CH^{d-1}(X \times C)$ such that $\beta = c_0^* \gamma - c_1^* \gamma$. Let $\pi_X : X \times C \rightarrow X$ and $\pi_C : X \times C \rightarrow C$ be the two projections.

The Gysin morphism ${}^t\pi : \mathbf{Z}(d)[2d] \rightarrow M(X)$ used in the definition of (6.4) extends trivially to give morphisms $M(d)[2d] \rightarrow M \otimes M(X)$ for

any $M \in \mathbf{DM}^{\text{eff}}$, which are clearly natural in M : this applies in particular to $M = M(C)$, giving a Gysin morphism ${}^t\pi_C : M(C)(d)[2d] \rightarrow M(X \times C)$ which induces a map

$$(\pi_C)_* : H^{2d+1}(X \times C, \mathbf{Z}(d+1)) \rightarrow H^1(C, \mathbf{Z}(1)).$$

The naturality of these Gysin morphisms then gives

$$\begin{aligned} \pi_*(\alpha \cdot \beta) &= \pi_*(\alpha \cdot (c_0^* \gamma - c_1^* \gamma)) \\ &= \pi_*(c_0^*(\pi_X^* \alpha \cdot \gamma) - c_1^*(\pi_X^* \alpha \cdot \gamma)) = (c_0^* - c_1^*)(\pi_C)_*(\pi_X^* \alpha \cdot \gamma). \end{aligned}$$

But $c_i^* : H^1(C, \mathbf{Z}(1)) \rightarrow H^1(F, \mathbf{Z}(1))$ is left inverse to $\pi'^* : H^1(F, \mathbf{Z}(1)) \rightarrow H^1(C, \mathbf{Z}(1))$ (where $\pi' : C \rightarrow \text{Spec } F$ is the structural map), which is an isomorphism since C is proper. Hence $c_0^* = c_1^*$ on $H^1(C, \mathbf{Z}(1))$, and the proof is complete. \square

7. PROOF OF THEOREM 4

Instead of Lewis' idea to use the complex Abel-Jacobi map, we use the l -adic Abel-Jacobi map in order to cover the case of arbitrary characteristic.

We may find a regular \mathbf{Z} -algebra R of finite type, a homomorphism $R \rightarrow F$, and a smooth projective scheme $p : \mathcal{X} \rightarrow \text{Spec } R$, such that $X = \mathcal{X} \otimes_R F$. By a \varinjlim argument, it suffices to show the theorem when F is the algebraic closure of the quotient field of R , and moreover, to show that the composition

$$H^1(\mathcal{X}, \mathcal{K}_2) \rightarrow H_{\text{ind}}^1(X, \mathcal{K}_2) \xrightarrow{\bar{\delta}} \text{Hom}(\text{Griff}_1(X), F^*)$$

has image killed by m .

Let l be a prime number different from $\text{char } F$. We may assume that l is invertible in R . We have l -adic regulator maps

$$H^1(\mathcal{X}, \mathcal{K}_2) \xrightarrow{c} H_{\text{ét}}^3(\mathcal{X}, \mathbf{Z}_l(2)), \quad H^{d-1}(\mathcal{X}, \mathcal{K}_{d-1}) \xrightarrow{c'} H_{\text{ét}}^{2d-2}(\mathcal{X}, \mathbf{Z}_l(d-1))$$

and two compatible pairings

$$(7.1) \quad H^1(\mathcal{X}, \mathcal{K}_2) \times H^{d-1}(\mathcal{X}, \mathcal{K}_{d-1}) \rightarrow H^d(\mathcal{X}, \mathcal{K}_{d+1}) \xrightarrow{p_*} H^0(R, \mathcal{K}_1) = R^*$$

$$(7.2) \quad H_{\text{ét}}^3(\mathcal{X}, \mathbf{Z}_l(2)) \times H_{\text{ét}}^{2d-2}(\mathcal{X}, \mathbf{Z}_l(d-1)) \rightarrow H_{\text{ét}}^{2d+1}(\mathcal{X}, \mathbf{Z}_l(d+1)) \xrightarrow{p_*} H_{\text{ét}}^1(R, \mathbf{Z}_l(1)).$$

The Leray spectral sequence for the projection p yields a filtration $F^r H_{\text{ét}}^*(\mathcal{X}, \mathbf{Z}_l(\bullet))$ on the l -adic cohomology of \mathcal{X} .

Let $H^{d-1}(\mathcal{X}, \mathcal{K}_{d-1})_0 = c'^{-1}(F^1 H_{\text{ét}}^{2d-2}(\mathcal{X}, \mathbf{Z}_l(d-1)))$ and $H^1(\mathcal{X}, \mathcal{K}_2)_0 = c^{-1}(F^1 H_{\text{ét}}^3(\mathcal{X}, \mathbf{Z}_l(2)))$.

7.1. Lemma. *The restriction of (7.1) to $H^1(\mathcal{X}, \mathcal{K}_2)_0 \times H^{d-1}(\mathcal{X}, \mathcal{K}_{d-1})_0$ has image in $R^*\{l'\}$, the subgroup of R^* of torsion prime to l .*

Proof. Since R is a finitely generated \mathbf{Z} -algebra, its group of units R^* is a finitely generated \mathbf{Z} -module, hence the Kummer map $R^* \otimes \mathbf{Z}_l \rightarrow H_{\text{ét}}^1(R, \mathbf{Z}_l(1))$ is injective; therefore the induced map $R^* \rightarrow H_{\text{ét}}^1(R, \mathbf{Z}_l(1))$ has finite kernel of cardinality prime to l . It therefore suffices to show that the restriction of (7.2) to

$$F^1 H_{\text{ét}}^3(\mathcal{X}, \mathbf{Z}_l(2)) \times F^1 H_{\text{ét}}^{2d-2}(\mathcal{X}, \mathbf{Z}_l(d-1))$$

is 0. By multiplicativity of the Leray spectral sequences, it suffices to show that $p_*(F^2 H_{\text{ét}}^{2d+1}(\mathcal{X}, \mathbf{Z}_l(d+1))) = 0$.

Since $\dim X = d$, we have $H_{\text{ét}}^0(R, H_{\text{ét}}^{2d+1}(X, \mathbf{Z}_l(d+1))) = 0$ and hence $H_{\text{ét}}^{2d+1}(\mathcal{X}, \mathbf{Z}_l(d+1)) = F^1 H_{\text{ét}}^{2d+1}(\mathcal{X}, \mathbf{Z}_l(d+1))$. The edge map

$$F^1 H_{\text{ét}}^{2d+1}(\mathcal{X}, \mathbf{Z}_l(d+1)) \rightarrow H_{\text{ét}}^1(R, H_{\text{ét}}^{2d}(X, \mathbf{Z}_l(d+1)))$$

coincides with the map p_* of (7.2) via the isomorphism

$$H_{\text{ét}}^{2d}(X, \mathbf{Z}_l(d+1)) \xrightarrow{p_*} H_{\text{ét}}^0(F, \mathbf{Z}_l(1)) = \mathbf{Z}_l(1).$$

This concludes the proof. \square

Passing to the \varinjlim in Lemma 7.1, we find that the pairing

$$H^1(X, \mathcal{K}_2)_0 \times CH^{d-1}(X)_0 \rightarrow F^*$$

has image in $F^*\{l'\}$.

7.2. Lemma. *The group $H^1(X, \mathcal{K}_2)/H^1(X, \mathcal{K}_2)_0$ is finite of exponent dividing m_l .*

Proof. It suffices to observe that the regulator map

$$H^1(X, \mathcal{K}_2) \rightarrow H_{\text{ét}}^3(X, \mathbf{Z}_l(2))$$

has finite image [7, Th. 2.2]. \square

Lemmas 7.1 and 7.2 show that the pairing $H^1(X, \mathcal{K}_2) \times CH^{d-1}(X) \rightarrow F^*$ has image in a group of roots of unity whose l -primary component is finite of exponent m_l for all primes $l \neq \text{char } F$. This completes the proof of Theorem 4.

8. QUESTIONS AND REMARKS

- (1) Does the conclusion of Proposition 3.2 remain true in general?
- (2) Can one give an *a priori*, concrete, description of the extension in Theorem 1 (iii)?
- (3) It is known that $\mathrm{Griff}_1(X) \otimes \mathbf{Q}$ (*resp* $\mathrm{Griff}_1(X)/l$ for some primes l) may be nonzero for some 3-folds X [10, 4]; these groups may not even be finite dimensional, *e.g.* [5, 20]. Can one find examples for which $\mathrm{Hom}(\mathrm{Griff}_1(X), \mathbf{Z}) \neq 0$?
- (4) To put the previous question in a wider context, let A be a torsion-free abelian group. Replacing \mathbb{G}_m by $\mathcal{F} = \mathbb{G}_m \otimes A$ in Theorem 1 yields the following computation (with same proofs):
 - (i) $R_{\mathrm{nr}}^0 \mathcal{F}(X) = F^* \otimes A$.
 - (ii) $R_{\mathrm{nr}}^1 \mathcal{F}(X) \xrightarrow{\sim} \mathrm{Pic}^\tau(X) \otimes A$.
 - (iii) There is a short exact sequence

$$(8.1) \quad 0 \rightarrow D^1(X) \otimes A \rightarrow R_{\mathrm{nr}}^2 \mathcal{F}(X) \rightarrow \mathrm{Hom}(\mathrm{Griff}_1(X), A) \rightarrow 0.$$

Taking $A = \mathbf{Q}$ we get examples, from the nontriviality of $\mathrm{Griff}_1(X) \otimes \mathbf{Q}$, where $R_{\mathrm{nr}}^2 \mathcal{F}(X)$ is not reduced to $D^1(X) \otimes A$. But, choosing X such that $\mathrm{Griff}_1(X) \otimes \mathbf{Q}$ is not finite dimensional and varying A among \mathbf{Q} -vector spaces, (8.1) also shows that *the functor $\mathcal{F} \mapsto R_{\mathrm{nr}}^2 \mathcal{F}$ does not commute with infinite direct sums*. This is all the more striking as R_{nr}^0 does commute with infinite direct sums, which is clear from Formula (1) in the introduction.

We don't know whether R_{nr}^1 commutes with infinite direct sums or not.

REFERENCES

- [1] J. Ayoub, L. Barbieri-Viale *1-motivic sheaves and the Albanese functor*, J. Pure Appl. Algebra **213** (2009), no. 5, 809–839.
- [2] L. Barbieri-Viale, B. Kahn *On the derived category of 1-motives*, to appear in Astérisque.
- [3] A. Beilinson, J. Bernstein, P. Deligne *Faisceaux pervers*, Astérisque **100**, 1984.
- [4] S. Bloch, H. Esnault *The coniveau filtration and non-divisibility for algebraic cycles*, Math. Ann. **304** (1996), 303–314.
- [5] H. Clemens *Homological equivalence, modulo algebraic equivalence, is not finitely generated* Publ. Math. IHÉS **58** (1983), 19–38.
- [6] J.-L. Colliot-Thélène *Unramified cohomology, birational invariants and the Gersten conjecture*, Proc. Sympos. Pure Math. **58**, Part 1, Amer. Math. Soc., Providence, RI, 1995.
- [7] J.-L. Colliot-Thélène, W. Raskind *\mathcal{K}_2 -cohomology and the second Chow group*, Math. Ann. **270** (1985), 165–199.
- [8] K. Coombes *The arithmetic of zero cycles on surfaces with geometric genus and irregularity zero*, Math. Ann. **291** (1991), 429–452.

- [9] O. Gabber *Sur la torsion dans la cohomologie l -adique d'une variété*, C. R. Acad. Sci. Paris **297** (1983), 179–182.
- [10] P.A. Griffiths *On the periods of certain rational integrals. I, II*, Ann. of Math. **90** (1969), 460–495 and 496–541.
- [11] A. Huber, B. Kahn *The slice filtration and mixed Tate motives*, Compositio Math. **142** (2006), 907–936.
- [12] B. Kahn *Motivic cohomology of smooth geometrically cellular varieties*, Proc. Symp. pure Math. **67** (1999), 149–174.
- [13] B. Kahn *The Brauer group and indecomposable $(2, 1)$ -cycles*, to appear in Compositio Math.
- [14] B. Kahn, R. Sujatha *Birational motives, II: triangulated birational motives*, <http://arxiv.org/abs/1506.08385>.
- [15] B. Kahn, T. Yamazaki *Voevodsky's motives and Weil reciprocity*, Duke Math. J. **162** (14) (2013), 2751–2796; erratum: **164** (2015), 2093–2098.
- [16] D. Lieberman *Numerical and homological equivalence of algebraic cycles on Hodge manifolds*, Amer. J. Math. **90** (1968), 366–374.
- [17] C. Mazza, V. Voevodsky, C. Weibel *Lecture notes on Motivic cohomology*, Clay Math. Monographs **2**, AMS, 2006.
- [18] J. P. Murre *Un résultat en théorie des cycles algébriques de codimension deux*, C. R. Acad. Sci. Paris **296** (1983), 981–984.
- [19] J. Riou *Théorie homotopique des S -schémas*, mémoire de DEA, Paris 7, 2002, <http://www.math.u-psud.fr/~riou/dea/dea.pdf>.
- [20] C. Schoen *Complex varieties for which the Chow group mod n is not finite*, J. Alg. Geom. **11** (2002), 41–100.
- [21] M. Spieß, T. Szamuely *On the Albanese map for smooth quasi-projective varieties*, Math. Ann. **325** (2003), 1–17.
- [22] V. Voevodsky *Cohomological theory of presheaves with transfers*, in E. Friedlander, A. Suslin, V. Voevodsky *Cycles, transfers and motivic cohomology theories*, Ann. Math. Studies **143**, Princeton University Press, 2000, 188–187.
- [23] V. Voevodsky *Triangulated categories of motives over a field*, in E. Friedlander, A. Suslin, V. Voevodsky *Cycles, transfers and motivic cohomology theories*, Ann. Math. Studies **143**, Princeton University Press, 2000, 188–238.
- [24] V. Voevodsky *Cancellation theorem*, Doc. Math. 2010, Extra volume: Andrei A. Suslin sixtieth birthday, 671–685.

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